

## Decentralized Pole Assignment for Interconnected Systems

Javad Lavaei and Amir G. Aghdam

**Abstract**—Given a general proper interconnected system, this paper aims to design a LTI decentralized controller to place the modes of the closed-loop system at pre-determined locations. To this end, it is first assumed that the structural graph of the system is strongly connected. Then, it is shown applying generic static local controllers to any number of subsystems will not introduce new decentralized fixed modes (DFM) in the resultant system, although it has fewer input-output stations compared to the original system. This means that if there are some subsystems whose control costs are highly dependent on the complexity of the control law, then generic static controllers can be applied to such subsystems, without changing the characteristics of the system in terms of the fixed modes. As a direct application of this result, in the case when the system has no DFMs, one can apply generic static controllers to all but one subsystem, and the resultant system will be controllable and observable through that subsystem. Now, a simple observer-based local controller corresponding to this subsystem can be designed to displace the modes of the entire system arbitrarily. Similar results can also be attained for a system whose structural graph is not strongly connected. It is worth mentioning that similar concepts are deployed in the literature for the special case of strictly proper systems, but as noted in the relevant papers, extension of the results to general proper systems is not trivial. This demonstrates the significance of the present work.

### I. INTRODUCTION

Every interconnected system can be envisaged as a system consisting of a number of interacting subsystems [1]. For such systems, it is often desired to impose certain constraints on structure of the controllers to be designed [2], [3]. These constraints originate from some practical issues, which can be broadly categorized as follows:

1. In many interconnected systems, the outputs of certain subsystems are not easily accessible to some other subsystems in the sense that such access requires costly data transmission. This is, for example, the case for the systems consisting of geographically distributed subsystems such as power networks.
2. In some situations, certain outputs of a system are completely inaccessible to some of the subsystems in specific time intervals. This, for instance, occurs in flight formation subject to the shadow phenomenon [4].
3. In the case when a large-scale system is composed of many subsystems, the computational complexity

associated with a centralized controller (i.e. an unconstrained controller) may be remarkably high. For such a system, it is normally desired to reduce the number of communication links.

These issues describe the demand for employing structurally constrained controllers. Note that a structurally constrained controller throughout this paper refers to a controller comprising a set of local controllers, which partially communicate with each other. Structurally constrained control design problem has been studied extensively over the last few decades, and has found applications in many practical systems such as power networks, flight formation and communication systems, to name only a few [1-9]. In the special case when each local controller observes only the output of its corresponding subsystem to construct the control input for that subsystem, the control structure is referred to as decentralized [10], [11], [12].

The primary issue in decentralized control theory is the stabilizability problem. In this regard, the notion of a decentralized fixed mode (DFM) was introduced in [10] to identify those modes of a system which are fixed with respect to every LTI decentralized controller. The work [13] proposed a method to characterize the DFMs of a *strictly proper* system in terms of the transfer function of the system. Later on, it was shown in [11] that a mode is a DFM if and only if several matrices satisfy a rank condition. Although this approach is appealing from the mathematical perspective, it is not computationally efficient because the number of matrices whose ranks should be checked grows exponentially by the number of subsystems. More recently, a simple combinatorial algorithm was proposed in [14] to identify the unrepeated DFMs of a system.

Since a system with unstable DFMs might be stabilizable by means of a non-LTI decentralized controller, the notion of quotient fixed modes (QFM) was introduced in [15] to find the modes of a system which are fixed with respect to any type of decentralized controller (nonlinear or time-varying). It is shown in [6] that the unrepeated DFMs of a system which are not QFMs can be eliminated via sampling, for almost all sampling periods. On the other hand, the work [7] proposes a simple method to eliminate the unwanted DFMs via a LTI controller by introducing the minimum number of information exchanges between the local controllers.

A structurally constrained controller is generally composed of partially interacting local controllers, rather than isolated local controllers. To address the relevant control design problems for any class of structurally constrained controllers, it is proved in [2] and [3] that designing a structurally constrained controller for a given system to

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Javad Lavaei is with the Department of Control and Dynamical Systems, California Institute of Technology, Pasadena, USA (email: lavaei@cds.caltech.edu).

Amir G. Aghdam is with the Department of Electrical and Computer Engineering, Concordia University, Montreal, Canada (email: aghdam@ece.concordia.ca).

achieve some rudimentary objectives (such as performance improvement or pole placement) is equivalent to designing a decentralized controller for an alternative system to attain the same objectives. As a direct consequence of this result, for investigating various properties of structurally constrained controllers, it suffices to restrict the attention merely on the particular class of decentralized controllers.

A very important problem associated with the decentralized control theory is pole assignability. Broadly speaking, decentralized pole placement is studied in the literature from two different perspectives to address the following problems:

- i) How can a LTI decentralized controller be designed to place the modes of the control system at prescribed locations?
- ii) What degrees should be considered for the local controllers in order for the decentralized pole-placement problem to be solvable?

Regarding problem (i), the work [16] supposes (with no loss of generality) that the structural graph of the system is strongly connected. Assuming the system has no DFMs, it recommends that generic static local controllers be applied to all the subsystems, except the last one. This work asserts that the resultant system is controllable and observable through its last subsystem. Now, an observer-based local controller (corresponding to the last subsystem) can be designed in order to displace the modes as desired. The main deficiency of this work is that it solely deals with *strictly proper* systems. However, the paper [11] argues that *non-strictly proper* systems play a key role in decentralized control. The paper [17] has further developed the idea presented in [16].

Problem (ii), on the other hand, has nothing to do with the controller design and mainly seeks the minimum degrees of the local controllers. Recently, this problem has become a focal point in the decentralized controller design. The objective of this problem is to present some bound on the degrees of the decentralized controllers by which the modes of a generic system can be arbitrarily assigned [18], [19], [20]. It is worth mentioning that since the method given in [16] exerts the entire control effort on only a single subsystem, it is often not feasible for large-scale interconnected systems, and in this case, employing a low-order decentralized controller to achieve the desired pole placement turns out to be vital.

This paper tackles the decentralized pole-placement problem for *general proper* systems. The system is first partitioned into a number of modified systems, such that each modified system encompasses some subsystems of the original system. Since these modified systems are in the hierarchical form [21], decentralized stabilizability of the overall system is equivalent to the decentralized stabilizability of all the modified systems separately. It is shown for a modified system that generic static local controllers can be applied to as many subsystems as desired with the property that the resultant modified system will have no new DFMs through the remaining open subsystems. This result can be simply exploited to straightforwardly design a decentralized controller to place the modes of the system at any pre-determined locations. It is interesting to note that the results

presented here are the generalization of those given in [17] for *strictly proper* systems. However, as it can be observed from the developments in the present work (and as it is pointed out in [11] for a similar problem), this generalization is not straightforward.

## II. MAIN RESULTS

Consider a linear time-invariant (LTI) interconnected system  $\mathcal{S}$  consisting of  $\nu$  subsystems  $S_1, S_2, \dots, S_\nu$ , represented by:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + \sum_{j=1}^{\nu} B_j u_j(t) \\ y_i(t) &= C_i x(t) + \sum_{j=1}^{\nu} D_{ij} u_j(t), \quad i \in \nu := \{1, 2, \dots, \nu\}\end{aligned}$$

where  $x(t) \in \mathbb{R}^n$  is the state, and  $u_i(t) \in \mathbb{R}^{m_i}$  and  $y_i(t) \in \mathbb{R}^{r_i}$ ,  $i \in \nu$ , are the input and the output of the  $i^{\text{th}}$  subsystem, respectively. Define the following matrices:

$$\begin{aligned}B &:= [B_1 \ \cdots \ B_\nu], \\ C &:= [C_1^T \ \cdots \ C_\nu^T]^T, \\ D &:= \begin{bmatrix} D_{11} & \cdots & D_{1\nu} \\ \vdots & \ddots & \vdots \\ D_{\nu 1} & \cdots & D_{\nu\nu} \end{bmatrix}\end{aligned}$$

Define also:

$$\begin{aligned}m &:= \sum_{i=1}^{\nu} m_i, \quad r := \sum_{i=1}^{\nu} r_i, \\ u(t) &:= [u_1(t)^T \ u_2(t)^T \ \cdots \ u_\nu(t)^T]^T \\ y(t) &:= [y_1(t)^T \ y_2(t)^T \ \cdots \ y_\nu(t)^T]^T\end{aligned}$$

Throughout the paper, decentralized controller for the system  $\mathcal{S}$  refers to the union of  $\nu$  local controllers, where the  $i^{\text{th}}$  local controller,  $i \in \nu$ , constructs the input  $u_i(t)$  only in terms of the local output  $y_i(t)$ . It is desired now to design a decentralized controller for the system  $\mathcal{S}$  so that all the modes of the closed-loop system are located at pre-determined locations, if possible. The following definitions and notations will prove to be essential in presenting the main results of the paper.

*Notation 1:* For any  $i \in \nu$ :

- let  $b_{i\alpha}$  denote the  $\alpha^{\text{th}}$  column of the matrix  $B_i$ , for any  $\alpha \in \{1, 2, \dots, m_i\}$ ;
- denote the  $\beta^{\text{th}}$  row of the matrix  $C_i$  with  $c_{i\beta}$ , for any  $\beta \in \{1, 2, \dots, r_i\}$ .

*Notation 2:* Denote with  $\mathbf{K}$  the space of all block-diagonal matrices whose  $i^{\text{th}}$  block entries,  $i \in \nu$ , are of dimension  $m_i \times r_i$ .

*Definition 1:* [11] A mode  $\lambda \in \text{sp}(A)$  is said to be a decentralized fixed mode (DFM) of the system  $\mathcal{S}$ , if it remains a mode of the closed-loop system under any arbitrary decentralized static output feedback. In other words, the mode  $\lambda \in \text{sp}(A)$  is a DFM of the system  $\mathcal{S}$  if:

$$\lambda \in \text{sp}(A + BK(I - DK)^{-1}C)$$

for any matrix  $K$  belonging to the set  $\mathbf{K}$ .

**Definition 2:** [8] Define the structural graph of the system  $\mathcal{S}$  as a digraph with  $\nu$  vertices which has a directed edge from the  $i^{\text{th}}$  vertex to the  $j^{\text{th}}$  vertex if and only if  $C_j(sI - A)^{-1}B_i + D_{ji} \neq 0$ , for any  $i, j \in \nu$ . The structural graph of the system  $\mathcal{S}$  is denoted by  $\mathcal{G}$ .

**Definition 3:** A digraph is called strongly connected if and only if there exists a directed path from any vertex to all other vertices of the graph.

Partition  $\mathcal{G}$  into the minimum number of strongly connected subgraphs denoted by  $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_l$ . It can be easily substantiated that if the system  $\mathcal{S}$  has a single input such that the transfer function from that input to the entire output of the system is equal to zero, then that input does not contribute to the control of the system, and hence can be eliminated due to its redundancy in the overall control operation. The same statement holds true for any single output of the system with the property that the transfer function from the entire input to that output is equal to zero. Thus, without loss of generality, assume that such inputs and outputs do not exist in the system  $\mathcal{S}$ .

#### A. A strongly connected system with no DFMs

Assume for now that the system  $\mathcal{S}$  has no DFMs, and that  $l = 1$ .

**Definition 4:** A subset  $V$  of  $\mathbb{R}^\mu$  is called a hypersurface if there exists a multivariate polynomial  $f(\omega)$  such that the set of its roots is identical to  $V$ .

Consider a hypersurface  $V$  in the  $\mu$ -dimensional space. Since the dimension of  $V$  is less than  $\mu$ , if a point is chosen in the  $\mu$ -dimensional space, it *almost always* does not belong to  $V$ . Moreover, any point in the  $\mu$ -dimensional space which does not lie on the hypersurface  $V$ , is referred to as a generic point. Note that generic points can be provided by using a random number generator.

**Theorem 1:** Consider an arbitrary matrix  $K \in \mathbf{K}$ , and apply the static decentralized controller  $u(t) = Ky(t)$  to the system  $\mathcal{S}$ . For any  $i, j \in \nu$ ,  $\alpha \in \{1, \dots, m_i\}$  and  $\beta \in \{1, \dots, r_j\}$ , the transfer function from the input  $u_{i\alpha}(t)$  to the output  $y_{j\beta}(t)$  in the resultant system is nonzero, unless the gain matrix  $K$  lies on a specific hypersurface.

**Proof:** It is shown in [17] that the transfer function matrix from  $u_i(t)$  to  $y_j(t)$  is almost always nonzero (for the case when  $D = 0$ ). However, the statement of this theorem is much stronger than the one in [17], in the sense that a matrix transfer function is proved to be almost always nonzero in [17], while it will be shown here that all entries of this matrix are generically nonzero. Using an approach similar to the one given in [17], this theorem will be proved. Nonetheless, it is essential first to show that there is a path from the input  $u_{i\alpha}(t)$  to the output  $y_{j\beta}(t)$  through the interconnections of the system  $\mathcal{S}$  and the local controllers of the decentralized controller  $u(t) = Ky(t)$ . This will be proved here under the assumption of  $D = 0$ , as the extension of this specific result to the case of general proper systems is straightforward.

By assumption, there is an output  $y_{i'\alpha'}(t)$  and an input  $u_{j'\beta'}(t)$  such that:

- i) The transfer function from  $u_{i\alpha}(t)$  to  $y_{i'\alpha'}(t)$  in the system  $\mathcal{S}$  is nonzero.
- ii) The transfer function from  $u_{j'\beta'}(t)$  to  $y_{j\beta}(t)$  in the system  $\mathcal{S}$  is nonzero.

Now, consider the subsystems  $S_{i'}$  and  $S_{j'}$ . Since the structural graph of the system is assumed to be strongly connected (i.e.  $l = 1$ ), one can conclude that there exist subsystems  $S_{\xi_1}, S_{\xi_2}, \dots, S_{\xi_\mu}$ , where  $S_{\xi_1} = S_{i'}$  and  $S_{\xi_\mu} = S_{j'}$ , with the following property:

$$C_{\xi_{\rho+1}}(sI - A)^{-1}B_{\xi_\rho} \neq 0, \quad \rho \in \{1, 2, \dots, \mu - 1\}$$

On the other hand, the above inequality implies that at least one of the scalar entries of the matrix  $C_{\xi_{\rho+1}}(sI - A)^{-1}B_{\xi_\rho}$  must be nonzero. As a result, there exist positive integers  $\zeta_1, \zeta_2, \dots, \zeta_\mu$  and  $\zeta'_1, \zeta'_2, \dots, \zeta'_\mu$  such that:

$$c_{\xi_{\rho+1}\zeta'_{\rho+1}}(sI - A)^{-1}b_{\xi_\rho\zeta_\rho} \neq 0, \quad \rho \in \{1, 2, \dots, \mu - 1\} \quad (1)$$

where  $\zeta'_1 = \alpha'$  and  $\zeta_\mu = \beta'$ . The inequalities given in (1) along with the above-mentioned assumptions (i) and (ii) result in the following path from the input  $u_{i\alpha}(t)$  to the output  $y_{j\beta}(t)$  (via the interconnections of the system  $\mathcal{S}$  and the static controller  $u(t) = Ky(t)$ ):

$$\begin{aligned} u_{i\alpha}(t) &\xrightarrow{T.F.} y_{\xi_1\zeta'_1}(t) \xrightarrow{L.C.} u_{\xi_1\zeta_1}(t) \xrightarrow{T.F.} y_{\xi_2\zeta'_2}(t) \\ &\xrightarrow{L.C.} u_{\xi_2\zeta_2}(t) \xrightarrow{T.F.} \dots \xrightarrow{L.C.} u_{\xi_\mu\zeta_\mu}(t) \xrightarrow{T.F.} y_{j\beta}(t) \end{aligned}$$

It is to be noted that *T.F.* and *L.C.* over the arrows in the above relation stand for *transfer function* and *local controller*, respectively, and are used to distinguish the two means of information transmission: interconnections of the system  $\mathcal{S}$ , and the decentralized controller  $u(t) = Ky(t)$ . The proof can be completed by using this important result and an approach analogous to the one given in [17]. The details are omitted here. ■

It is shown in [2] that there exist two matrices  $\Phi_1$  and  $\Phi_2$  with binary entries such that for any matrix  $K \in \mathbf{K}$ , the following relation holds:

$$K = \Phi_1 \bar{K} \Phi_2$$

where  $\bar{K}$  is a purely diagonal matrix obtained from  $K$  by putting its nonzero entries successively on the main diagonal of  $\bar{K}$ . Define now  $\bar{\mathcal{S}}$  as a system with the state-space representation given below:

$$\begin{aligned} \dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t) \\ \bar{y}(t) &= \bar{C}\bar{x}(t) + \bar{D}\bar{u}(t) \end{aligned}$$

where

$$\bar{A} = A, \quad \bar{B} = B\Phi_1, \quad \bar{C} = \Phi_2C, \quad \bar{D} = \Phi_2D\Phi_1$$

The following lemma is asserted in [2] for the system  $\bar{\mathcal{S}}$ .

**Lemma 1:** Consider a (dynamic) decentralized controller for the system  $\mathcal{S}$  with the transfer function matrix  $K(s)$ . Assume that  $\bar{K}(s)$  is derived from  $K(s)$  in the same way that  $\bar{K}$  was obtained from  $K$ . The modes of the system  $\mathcal{S}$  under the decentralized controller  $K(s)$  are tantamount to those of the system  $\bar{\mathcal{S}}$  under the decentralized controller  $\bar{K}(s)$ .

The advantage of forming the system  $\bar{\mathcal{S}}$  is that, unlike  $\mathcal{S}$ , all of its control agents are single-input single-output (SISO), while it preserves the pole-assignability property of the original system (according to Lemma 1). This facilitates the handling of the underlying problem. It is straightforward to verify that the system  $\bar{\mathcal{S}}$  has  $\bar{\nu}$  SISO subsystems, where  $\bar{\nu} := \sum_{i=1}^{\nu} m_i r_i$ .

*Notation 3:* For any  $i, j \in \{1, 2, \dots, \bar{\nu}\}$ :

- denote the input and the output of the  $i^{\text{th}}$  SISO subsystem of  $\bar{\mathcal{S}}$  with  $\bar{u}_i(t)$  and  $\bar{y}_i(t)$ , respectively;
- denote the  $i^{\text{th}}$  column and the  $i^{\text{th}}$  row of the matrices  $\bar{B}$  and  $\bar{C}$  with  $\bar{b}_i$  and  $\bar{c}_i$ , respectively;
- denote the  $(i, j)$  entry of the matrix  $\bar{D}$  with  $\bar{d}_{ij}$ .

*Corollary 1:* Consider an arbitrary matrix  $\bar{K}$  of proper dimension, and apply the static decentralized controller  $\bar{u}(t) = \bar{K}\bar{y}(t)$  to the system  $\bar{\mathcal{S}}$ . For any  $\zeta_1, \zeta_2 \in \{1, 2, \dots, \bar{\nu}\}$ , the transfer function from the input  $\bar{u}_{\zeta_1}(t)$  to the output  $\bar{y}_{\zeta_2}(t)$  in the resultant system is nonzero, unless the gain matrix  $\bar{K}$  lies on a specific hypersurface.

*Proof:* The proof follows immediately from Theorem 1 and on noting the relationship between the inputs and outputs of the systems  $\mathcal{S}$  and  $\bar{\mathcal{S}}$  [3]. ■

Corollary 1 states that for almost all matrices  $\bar{K}$ , the system  $\bar{\mathcal{S}}$  under the decentralized controller  $\bar{u}(t) = \bar{K}\bar{y}(t)$  has a transfer function matrix whose entries are all nonzero scalar functions. Therefore, assume that none of the entries of the transfer function matrix of the system  $\bar{\mathcal{S}}$  is identical to zero (this can be achieved by applying a generic static decentralized controller to the system, if necessary). Since it is assumed that the system  $\mathcal{S}$  has no DFMs, it can be inferred from Lemma 1 that none of the modes of the system  $\bar{\mathcal{S}}$  is a DFM either.

Consider an arbitrary scalar  $g$ . Let  $\bar{\mathcal{S}}(g)$  denote a system with the following properties:

- It is formed by applying the static controller  $\bar{u}_1(t) = g\bar{y}_1(t)$  to the system  $\bar{\mathcal{S}}$ .
- Its outputs and inputs are  $\bar{y}_2(t), \bar{y}_3(t), \dots, \bar{y}_{\bar{\nu}}(t)$  and  $\bar{u}_2(t), \bar{u}_3(t), \dots, \bar{u}_{\bar{\nu}}(t)$ , respectively.

Represent the LTI model of the system  $\bar{\mathcal{S}}(g)$  as:

$$\begin{aligned} \dot{\tilde{x}}(t) &= \tilde{A}(g)\tilde{x}(t) + \sum_{j=2}^{\bar{\nu}} \tilde{b}_j(g)\bar{u}_j(t) \\ \bar{y}_i(t) &= \tilde{c}_i(g)\tilde{x}(t) + \sum_{j=2}^{\bar{\nu}} \tilde{d}_{ij}(g)\bar{u}_j(t), \quad i \in \{2, \dots, \bar{\nu}\} \end{aligned}$$

At this point, the objective is to prove that the system  $\bar{\mathcal{S}}(g)$  with  $\bar{\nu} - 1$  SISO subsystems has no DFMs for almost all values of  $g$ . This is carried out in the sequel.

*Theorem 2:* Assume that there exists a diagonal matrix  $\bar{K}^* \in \mathbb{R}^{\bar{\nu} \times \bar{\nu}}$  such that the modes of the system  $\bar{\mathcal{S}}$  under the decentralized controller  $\bar{u}(t) = \bar{K}^*\bar{y}(t)$  are all distinct. Then, for almost all diagonal matrices  $\bar{K} \in \mathbb{R}^{\bar{\nu} \times \bar{\nu}}$ , the modes of the system  $\bar{\mathcal{S}}$  are also distinct under the decentralized controller  $\bar{u}(t) = \bar{K}\bar{y}(t)$ .

*Proof:* The proof is omitted here due to space restrictions. ■

*Corollary 2:* For almost all diagonal matrices  $\bar{K} \in \mathbb{R}^{\bar{\nu} \times \bar{\nu}}$ , the modes of the system  $\bar{\mathcal{S}}$  are distinct under the decentralized controller  $\bar{u}(t) = \bar{K}\bar{y}(t)$ .

*Proof:* As a result of Theorem 2, it suffices to show that there exists a diagonal matrix  $\bar{K}^* \in \mathbb{R}^{\bar{\nu} \times \bar{\nu}}$  such that the modes of the system  $\bar{\mathcal{S}}$  under the decentralized controller  $\bar{u}(t) = \bar{K}^*\bar{y}(t)$  are all distinct.

To prove the above-mentioned statement, assume that  $\sigma$  is a mode of the system  $\bar{\mathcal{S}}$  with a multiplicity of greater than one. For any  $j \in \{1, 2, \dots, \bar{\nu}\}$ , let  $\lambda_j$  be constructed as follows:

*Freeze the inputs and outputs of SISO subsystems  $j + 1, j + 2, \dots, \bar{\nu}$  of the system  $\bar{\mathcal{S}}$ . The resultant system has only  $j$  subsystems. Denote the set of the DFMs of this new system with  $\lambda_j$ .*

Since the system  $\bar{\mathcal{S}}$  has no DFMs,  $\lambda_{\bar{\nu}}$  is an empty set. Due to a manifest property of DFM,  $\lambda_j$  is a subset of  $\lambda_{j-1}$ , for any  $j \in \{2, 3, \dots, \bar{\nu}\}$ . Denote with  $\beta_i$  the number of times that  $\sigma$  appears in  $\lambda_i$ , for any  $i \in \{1, \dots, \bar{\nu}\}$ . It is evident that  $\beta_1 \geq \beta_2 \geq \dots \geq \beta_{\bar{\nu}} = 0$ . Let  $i$  be the smallest positive integer such that  $\beta_i$  is strictly less than  $\beta_{i-1}$ . It can be concluded from [11] that for almost all scalars  $g_1, g_2, \dots, g_{i-1}$ , the mode  $\sigma$  is observable and controllable  $\beta_{i-1} - \beta_i$  times through the  $i^{\text{th}}$  subsystem of the system obtained from  $\bar{\mathcal{S}}$  on applying the local controllers  $\bar{u}_j(t) = g_j\bar{y}_j(t)$ ,  $\forall j \in \{1, 2, \dots, i-1\}$ . Note that a generic choice of  $g_1, g_2, \dots, g_{i-1}$  would shift some modes of the system  $\bar{\mathcal{S}}$  to disjoint locations, which are also distinct from  $\sigma$  (this can be shown in line with the proof of Theorem 2). Now, consider the system  $\bar{\mathcal{S}}$  under the local controllers  $\bar{u}_j(t) = g_j\bar{y}_j(t)$ ,  $\forall j \in \{1, 2, \dots, i-1\}$ . Find the root locus trajectory corresponding to the single input and the single output of the  $i^{\text{th}}$  SISO subsystem of the resultant system. It is straightforward to shown that  $\beta_{i-1} - \beta_i$  branches originate from the point  $\sigma$  and diverge at different angles. This means that a proper static controller for this subsystem will decrease the multiplicity of the repeated mode  $\sigma$  by  $\beta_{i-1} - \beta_i$ . Now, one can consider the second smallest value of  $i$  for which  $\beta_i$  is strictly less than  $\beta_{i-1}$  and make the same argument for the new partially-closed-loop system. This will eventually lead to a static decentralized controller  $\bar{K}^*$  which provides distinct closed-loop modes. ■

*Theorem 3:* Assume that  $\sigma$  is a mode of the system  $\bar{\mathcal{S}}(g)$ . Then,  $\sigma$  is not a DFM of the system  $\bar{\mathcal{S}}(g)$ , unless the scalar  $g$  pertains to a specific finite set.

*Proof:* There are two possibilities for the mode  $\sigma$  as follows:

- It is a mode of the system  $\bar{\mathcal{S}}$  as well.
- It is distinct from all modes of the system  $\bar{\mathcal{S}}$ .

First, consider case (i). It is straightforward to show that  $\sigma$  is not a DFM of the system  $\bar{\mathcal{S}}(g)$  if and only if there exists a diagonal matrix  $\bar{K} \in \mathbb{R}^{(\bar{\nu}-1) \times (\bar{\nu}-1)}$  such that:

$$\sigma \notin \text{sp} \left( \bar{A} + \bar{B} \begin{bmatrix} g & 0 \\ 0 & \bar{K} \end{bmatrix} \left( I - \bar{D} \begin{bmatrix} g & 0 \\ 0 & \bar{K} \end{bmatrix} \right)^{-1} \bar{C} \right) \quad (2)$$

On the other hand, since the system  $\bar{\mathcal{S}}$  is assumed to have no DFMs and  $\sigma$  is a mode of this system, it can be deduced

from [11] that:

$$\sigma \notin \text{sp} \left( \bar{A} + \bar{B} \begin{bmatrix} e & 0 \\ 0 & \tilde{K} \end{bmatrix} \left( I - \bar{D} \begin{bmatrix} e & 0 \\ 0 & \tilde{K} \end{bmatrix} \right)^{-1} \bar{C} \right) \quad (3)$$

for almost all scalar values  $e$  and diagonal matrices  $\tilde{K} \in \mathbb{R}^{(\bar{\nu}-1) \times (\bar{\nu}-1)}$ . This result can be equivalently stated as *for almost all values of  $e$ , there exists at least one diagonal matrix  $\tilde{K} \in \mathbb{R}^{(\bar{\nu}-1) \times (\bar{\nu}-1)}$  for which the relation (3) holds.* The proof of case (i) follows now from this statement and the relation between (3) and (2).

Now, consider case (ii). One can easily conclude from Corollary 2 that the multiplicity of  $\sigma$  as a mode of the system  $\bar{\mathcal{S}}(g)$  is equal to 1 for almost all values of  $g$ . Suppose that the value of  $g$  is chosen such that  $\sigma$  is an unrepeated mode (i.e.  $g$  is a generic value). It is desired to prove that the mode  $\sigma$  of the system  $\bar{\mathcal{S}}(g)$  is almost always controllable and observable through its first subsystem (i.e. the input  $\bar{u}_2(t)$  and the output  $\bar{y}_2(t)$ ). To prove that the mode  $\sigma$  is controllable from the input  $\bar{u}_2(t)$  of the system  $\bar{\mathcal{S}}(g)$  for generic values of  $g$ , it suffices to show that the matrix:

$$(sI - \bar{A}(g))^{-1} \times \bar{b}_2(g)$$

has at least one infinite entry at  $s = \sigma$  (note that  $\sigma$  is an unrepeated mode of  $\bar{\mathcal{S}}(g)$ ). To this end, one can write:

$$\begin{aligned} (sI - \bar{A}(g))^{-1} \bar{b}_2(g) &= (sI - \bar{A} - \bar{b}_1 g (1 - \bar{d}_{11} g)^{-1} \bar{c}_1)^{-1} \\ &\quad \times (\bar{b}_2 + \bar{b}_1 g (1 - \bar{d}_{11} g)^{-1} \bar{d}_{12}) \\ &= (sI - \bar{A})^{-1} \bar{b}_1 g (1 - \bar{d}_{11} g)^{-1} \\ &\quad \times (1 - \bar{c}_1 (sI - \bar{A})^{-1} \bar{b}_1 g (1 - \bar{d}_{11} g)^{-1})^{-1} \\ &\quad \times (\bar{c}_1 (sI - \bar{A})^{-1} \bar{b}_2 + \bar{d}_{12}) \\ &\quad + (sI - \bar{A})^{-1} \bar{b}_2 \\ &= \frac{g}{1 - \bar{d}_{11} g - \bar{c}_1 (sI - \bar{A})^{-1} \bar{b}_1 g} h_1(s) h_2(s) \\ &\quad + (s - \bar{A})^{-1} \bar{b}_2 \end{aligned} \quad (4)$$

where

$$h_1(s) = (sI - \bar{A})^{-1} \bar{b}_1, \quad h_2(s) = \bar{c}_1 (sI - \bar{A})^{-1} \bar{b}_2 + \bar{d}_{12}$$

Since  $\sigma$  is not an eigenvalue of the matrix  $\bar{A}$  (as stated in case (ii)), it can be easily verified that:

$$1 - \bar{d}_{11} g - \bar{c}_1 (\sigma I - \bar{A})^{-1} \bar{b}_1 g = 0 \quad (5)$$

On the other hand, the term  $(s - \bar{A})^{-1} \bar{b}_2$  is finite at  $s = \sigma$  (from the condition in case (ii)). Assume that  $\sigma$  is uncontrollable from the input  $\bar{u}_2(t)$  of the system  $\bar{\mathcal{S}}(g)$ , for some  $g$ . It can be concluded from (4) and (5) that  $h_1(\sigma) h_2(\sigma)$  is a zero vector. It is obvious that  $h_1(s)$  is not identical to zero for all values of  $s$ . Moreover,  $h_2(s)$  is the transfer function from the second input of  $\bar{\mathcal{S}}$  to its first output. From the assumption following Corollary 1, the transfer function  $h_2(s)$  is not identical to zero. As a result,  $h_1(s) h_2(s)$  is a vector with rational entries. Since  $\sigma$  is a zero of this rational function, it is located on a hypersurface. On the other hand, the values of  $g$  for which the system  $\bar{\mathcal{S}}(g)$  has a mode  $\sigma$  on that specific hypersurface lie on another hypersurface. This

asserts that the mode  $\sigma$  is controllable from the input  $\bar{u}_2(t)$  of the system  $\bar{\mathcal{S}}(g)$  for almost all values of  $g$ . Likewise, it can be shown that the mode  $\sigma$  is observable from the output  $\bar{y}_2(t)$  of the system  $\bar{\mathcal{S}}(g)$  for almost all values of  $g$ . Since it is shown that the mode  $\sigma$  of the system  $\bar{\mathcal{S}}(g)$  is almost always controllable and observable through its first subsystem,  $\sigma$  is not a DFM of the system for generic values of  $g$ . ■

**Theorem 4:** Apply the static controller  $u_i(t) = K_i y_i(t)$  to the system  $\mathcal{S}$ , for any  $i \in \{1, 2, \dots, \nu - 1\}$ , where  $K_1, K_2, \dots, K_{\nu-1}$  are arbitrary constant matrices of proper dimensions. The resultant system is controllable and observable through the input and from the output of its last subsystem (i.e.  $u_\nu(t)$  and  $y_\nu(t)$ ), unless the matrices  $K_1, K_2, \dots, K_{\nu-1}$  are located on a certain hypersurface.

*Proof:* The proof is omitted here due to space restrictions. ■

**Remark 1:** Theorem 4 proposes static controllers to be applied to all but one subsystem. Thus, the system is mainly to be controlled from a single subsystem. This may encumber the corresponding control function in the case when the system consists of a large number of subsystems. In order to alleviate this problem, one can apply generic static controllers to fewer number of subsystems. It can be shown in line with the proof of Theorem 4 that the resultant system in this case will still have no fixed modes through the inputs and the outputs of the remaining subsystems. Now, one of the existing pole-placement approaches, e.g. the one proposed in [19], can be deployed to design dynamic local controllers for the uncontrolled subsystems to place the closed-loop modes at the desired locations.

## B. The general case for the system $\mathcal{S}$

The results presented so far are attained based on the assumptions that  $l = 1$  and the system  $\mathcal{S}$  has no DFMs. It is desired now to obviate these assumptions.

As pointed out in [16], there exists a coordination under which the state-space representation of the system comprises  $l$  modified systems which are physically in the hierarchical form; i.e., no information is transmitted from the modified system  $i$  to the modified system  $j$  for any  $i, j \in \{1, 2, \dots, l\}$ ,  $i > j$ . This implies that the decentralized pole placement can be equivalently solved for the modified systems  $1, 2, \dots, l$  separately. In other words, the decentralized pole-placement problem for the system  $\mathcal{S}$  can be accomplished by solving  $l$  decentralized pole-placement problems such that the  $i^{\text{th}}$  problem is for the  $i^{\text{th}}$  modified system with the strongly-connected graph  $\mathcal{G}_i$ , for any  $i \in \{1, 2, \dots, l\}$ .

The following algorithm takes advantage of the above discussion and Theorem 4, and proposes a systematic method to place the modes of the system at any pre-determined locations denoted by the set  $\rho$  of size  $n$ , by means of a LTI decentralized controller. It is notable that the set  $\rho$  may have repeated elements.

**Algorithm 1:**

- Step 1: Obtain the structural graph of the system  $\mathcal{S}$  and partition it into the minimum number of strongly

connected subgraphs. Denote this minimum number with  $l$ .

- Step 2: Decompose the system  $\mathcal{S}$  into  $l$  new systems  $\tilde{S}_1, \tilde{S}_2, \dots, \tilde{S}_l$  (with strongly connected subgraphs), which are in the hierarchical form; i.e., there is no interconnection from  $\tilde{S}_i$  to  $\tilde{S}_j$ , for any  $i, j \in \{1, 2, \dots, l\}$ ,  $i > j$ . This decomposition can be derived from the structural graph of the system  $\mathcal{S}$ , by exploiting the technique given in [16].
- Step 3: For any  $i \in \{1, 2, \dots, l\}$ , obtain the set of the DFMs of the system  $\tilde{S}_i$  and denote it with  $\gamma_i$ .
- Step 4: Check whether the multiplicity of every element of the set  $\rho$  is greater than or equal to the number of times that the element appears in the sets  $\gamma_1, \gamma_2, \dots, \gamma_l$  (including its repetition). If yes, proceed to the next step; otherwise, the decentralized pole placement for the desired locations is not feasible and the algorithm should halt here.
- Step 5: Consider  $\rho_1 = \rho$ . For  $i = 1$  to  $l$ , carry out the following procedure:
  1. Apply generic static local controllers to all subsystems of  $\tilde{S}_i$ , except the last one. Now, design a Luenberger observer and a static controller for the last subsystem of the resultant system to place the non-DFM modes of the system at any locations belonging to  $\rho_i$  and disjoint from the elements of  $\gamma_{i+1}, \gamma_{i+2}, \dots, \gamma_l$  (there may exist several choices here).
  2. Remove all modes of the new closed-loop system from  $\rho_i$  and denote the resultant set with  $\rho_{i+1}$ .
- Step 6: All the modes of the system  $\mathcal{S}$  are now at the desired locations given by the set  $\rho$ .

### III. NUMERICAL EXAMPLE

Consider a system  $\mathcal{S}$  consisting of four SISO subsystems with the following decoupled state-space matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 6 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0 \\ 7 \\ 9 \\ -5 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 \\ 0 \\ 8 \\ 7 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 3 \end{bmatrix}^T$$

$$C_2 = \begin{bmatrix} 0 \\ -6 \\ 0 \\ 8 \end{bmatrix}^T, \quad C_3 = \begin{bmatrix} 0 \\ 4 \\ 0 \\ -9 \end{bmatrix}^T, \quad C_4 = \begin{bmatrix} 5 \\ 1 \\ 0 \\ 7 \end{bmatrix}^T$$

$$D_{11} = -5, \quad D_{12} = 10, \quad D_{13} = 27, \quad D_{14} = 23,$$

$$D_{21} = 32, \quad D_{22} = 60, \quad D_{23} = -3, \quad D_{24} = 56/3,$$

$$D_{31} = -25, \quad D_{32} = -62, \quad D_{33} = 43, \quad D_{34} = -21,$$

$$D_{41} = -4.5, \quad D_{42} = 40, \quad D_{43} = 16, \quad D_{44} = 7,$$

It is desired to place the modes of the system arbitrarily, say at  $\lambda_1, \lambda_2, \lambda_3$  and  $\lambda_4$ . Assume that due to the practi-

cal constraints, using a structurally constrained controller is preferred. Commence with a decentralized control law  $U(s) = K(s)Y(s)$ , where  $K(s)$  is the transfer function of the controller being sought, which is represented as:

$$K(s) = \begin{bmatrix} K_{11}(s) & 0 & 0 & 0 \\ 0 & K_{22}(s) & 0 & 0 \\ 0 & 0 & K_{33}(s) & 0 \\ 0 & 0 & 0 & K_{44}(s) \end{bmatrix}$$

As discussed in [7], the system has two DFMs 1 and 3, and since they are unstable, there is no LTI decentralized controller to stabilize the system. Moreover, the work [7] has shown that if a single communication link is added between any two isolated local controllers, the system will still have at least one unstable fixed mode. The work [7] suggests adding two communication links between the local controllers in order to achieve stability. In this case, the structurally constrained controller  $K(s)$  would have the following form [7]:

$$K(s) = \begin{bmatrix} K_{11}(s) & 0 & 0 & K_{14}(s) \\ 0 & K_{22}(s) & 0 & 0 \\ K_{31}(s) & 0 & K_{33}(s) & 0 \\ 0 & 0 & 0 & K_{44}(s) \end{bmatrix} \quad (6)$$

Therefore, the objective now reduces to designing a dynamic controller of the form (6) so that the modes of the closed-loop system are placed at  $\lambda_1, \lambda_2, \lambda_3$  and  $\lambda_4$ . To this end, let this controller be transformed into the conventional form by means of the technique presented in [2]. Define the matrix  $\hat{K}(s)$  as:

$$\begin{bmatrix} K_{11}(s) & K_{14}(s) & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{22}(s) & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{31}(s) & K_{33}(s) & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{44}(s) \end{bmatrix}$$

and also the system  $\hat{\mathcal{S}}$  as:

$$\begin{aligned} \hat{\dot{x}}(t) &= \hat{A}\hat{x}(t) + \hat{B}\hat{u}(t) \\ \hat{y}(t) &= \hat{C}\hat{x}(t) + \hat{D}\hat{u}(t) \end{aligned}$$

where  $\hat{A} = A, \hat{B} = B$ , and

$$\begin{aligned} \hat{C} &= [C_1^T \quad C_4^T \quad C_2^T \quad C_1^T \quad C_3^T \quad C_4^T]^T, \\ \hat{D} &= [D_1^T \quad D_4^T \quad D_2^T \quad D_1^T \quad D_3^T \quad D_4^T]^T \end{aligned}$$

Note that  $D_i$  represents the  $i^{\text{th}}$  row of the matrix  $D$ , for any  $i \in \{1, 2, 3, 4\}$ . As stated in [2], the system  $\mathcal{S}$  under the structurally constrained controller  $U(s) = K(s)Y(s)$  possesses the same modes as the system  $\hat{\mathcal{S}}$  under the decentralized controller  $\hat{U}(s) = \hat{K}(s)\hat{Y}(s)$ . Since the matrix  $\hat{K}(s)$  (whose elements are to be designed) is block diagonal, the method proposed in this paper can be applied to accomplish the desirable pole placement. It is worth mentioning that the subsystems of  $\hat{\mathcal{S}}$  are all single input and have 2, 1, 2 and 1 output(s), respectively. It can be easily verified that the structural graph of  $\hat{\mathcal{S}}$  is strongly connected (i.e.  $l = 1$ ). Hence, step 2 of Algorithm 1 can be bypassed here. At this

point, consider the local controllers of subsystems 2, 3 and 4 of the system  $\hat{S}$  as generic static controllers; for instance:

$$K_{22}(s) = K_{31}(s) = K_{33}(s) = K_{44}(s) = 1$$

Let these static controllers be applied to the system  $\hat{S}$ . The resultant system through its first subsystem will have the following model:

$$\begin{aligned} \dot{\tilde{x}}(t) &= \begin{bmatrix} 1.0000 & 0 & 0 & 0 \\ -1.1172 & 1.0381 & -0.1852 & -0.6720 \\ 2.3553 & 2.1995 & 1.9454 & 0.4655 \\ 2.7451 & 3.3367 & -0.3138 & 3.9694 \end{bmatrix} \tilde{x}(t) \\ &+ \begin{bmatrix} 3.0000 \\ 8.8514 \\ -11.8656 \\ -16.4950 \end{bmatrix} \ddot{u}(t) \\ \ddot{y}(t) &= \begin{bmatrix} 5.2213 & 7.8119 & 1.2065 & 2.5562 \\ 0.4284 & 0.4177 & -0.0932 & 0.1106 \end{bmatrix} \tilde{x}(t) \\ &+ \begin{bmatrix} -36.5234 \\ -2.2028 \end{bmatrix} \ddot{u}(t) \end{aligned} \quad (7)$$

Now, a controller in the form of  $\ddot{U}(s) = [K_{11}(s) \ K_{14}(s)] \ddot{Y}(s)$  should be designed for the system given in (7) such that the modes of the closed-loop system are placed at  $\lambda_1, \lambda_2, \lambda_3$  and  $\lambda_4$ . It is straightforward to check that the controllability and observability matrices associated with the system (7) are both full-rank. This is in accordance with the result of Theorem 4 for this example. Thus, the pole-placement procedure can be carried out by designing a simple static state feedback controller and a Luenberger observer. The transfer function of the obtained observer-based controller can be equated to  $[K_{11}(s) \ K_{14}(s)]$  for obtaining the controller components  $K_{11}(s)$  and  $K_{14}(s)$ . For any given  $\lambda_1, \lambda_2, \lambda_3$  and  $\lambda_4$ , this pole-placement problem can be straightforwardly treated and the corresponding local controllers will be derived.

#### IV. CONCLUSIONS

This paper is concerned with decentralized pole assignment of general proper interconnected systems. The system is first decomposed into a number of modified systems based on its structural graph. It is shown that designing a decentralized controller for the system to achieve arbitrary pole placement is tantamount to designing a set of decentralized controllers corresponding to different modified systems such that each decentralized controller places a portion of the modes at pre-determined locations. As a result, the problem of designing a decentralized controller for a modified system to accomplish pole placement is treated in order to solve the original problem. It is shown that on applying generic static local controllers to all the subsystems of this modified system, except the last one, the resultant system will have no new decentralized fixed mode (DFM) through its last subsystem. This means that if the modified system has no DFMs, the resultant system will be controllable and observable through its last control agent. An observer-based controller can then

be designed for the last subsystem in order to place the modes at the desired locations. It is to be noted that in the special case when the system is *strictly proper*, similar results have already been presented in the literature; however, their extension to the general case is quite challenging as noted in the relevant papers.

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